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The high wavenumber instabilities of a Stokes wave

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A stability analysis for high wavenumber perturbations of a Stokes wave of wavenumber k_1 and slope ϵ is presented. Except for a correction term the governing equation is shown to be of Hill's type. The analysis predicts instability at wavenumbers $k_2 = \frac{1}{4}(m+1)^2 k_1$. The two lowest and strongest instabilities are the Benjamin–Feir instability at $m = 1$, and the quartet resonance at $m = 2$. Both are incorrectly treated by the present method. For $m \geqslant 3$ the analysis should be asymptotically ($\epsilon \to 0$) correct, yielding instability $O(\epsilon^m)$ due to m-fold Bragg-scattering. The non-resonant perturbations behave as predicted by WKBJ theory. The instability is too weak for experimental detection; numerical tests should be possible, but are not available at present.

1. Introduction

We shall consider the interaction of short gravity waves with wavenumber k_2 riding on a Stokes wave with wavenumber k_1. All motion shall be irrotational on an ocean of infinite depth and horizontal extension; surface tension will be neglected. Only the case k_2 parallel to k_1 will be studied.

The Stokes wave shall have small but finite slope $\epsilon = k_1 A_1$ (we have $\epsilon \lesssim 0{\cdot}1$ in mind); the short wave shall have infinitesimal non-dimensional amplitude $\epsilon_2 = k_1 A_2$.

We present a stability analysis for the short waves, linear in ϵ_2, but formally retain all powers in ϵ. Due to subsequent expansions in ϵ the theory is nevertheless only asymptotic ($\epsilon \to 0$). The relevant parameters of the problem are ϵ and $\alpha^2 = k_1/k_2$ and our approximations require $\alpha < 0{\cdot}5$.

The results are identical with WKBJ predictions (Garrett & Smith 1976) unless $k_2 = \frac{1}{4}(m+1)^2 k_1$, where m-fold Bragg scattering occurs, since

$$\tfrac{1}{4}(m+1)^2 - \tfrac{1}{4}(m-1)^2 = m \tag{1.1}$$

and

$$\tfrac{1}{2}(m+1) + \tfrac{1}{2}(m-1) = m. \tag{1.2}$$

The resulting instability is $O(\epsilon^m)$.

At $m = 1$ and $m = 2$ the Benjamin–Feir instability (Benjamin & Feir 1967; Benjamin 1967) and the quartet resonance (Phillips 1960, 1961; Bretherton 1964; Hasselmann 1962) are predicted but details are qualitatively ($m = 1$) or quantitatively ($m = 2$) incorrect.

Since the instabilities for $m \geqslant 3$ and $\epsilon < 0{\cdot}3$ are too weak for experimental detection the main purpose of this paper is to guide numerical calculations (Longuet-Higgins 1978*a*, *b*) to the most promising areas in the ϵ, α plane. We scale lengths with k_1^{-1} and time with $\sigma_1^{-1} = (k_1 g)^{-\frac{1}{2}}$, thus g is scaled to unity, $g = 1$.

R_D and R_s are the frames of reference in which either the water at depth (R_D) or the Stokes wave is at rest. Frequencies are Ω in R_D and ω in R_s. The Stokes wave propagates 'forwards' in R_D; y is positive upwards.

2. Formulation and reduction of the problem to standard form

In R_s the wave height ζ_1 of the Stokes wave is given as

$$\zeta_1 = \epsilon \cos x + \tfrac{1}{2}\epsilon^2 \cos 2x + O(\epsilon^3) \tag{2.1}$$

and for the potential ϕ_1 and the streamfunction ψ_1 see Lamb (1879, § 250). With the phase velocity $\beta = (1+\tfrac{1}{2}\epsilon^2)+O(\epsilon^3)$ we introduce orthogonal co-ordinates

$$\xi = -\phi_1/\beta \tag{2.2}$$

and

$$\eta = \psi_1/\beta \tag{2.3}$$

so that $\varphi_1(\xi,\eta) = \phi_1(x,y) = -\beta\xi$ and the surface is $\eta = 0$. An infinitesimal perturbation is then described by

$$\eta - \epsilon_2 w_2(\xi, t) = 0 \tag{2.4}$$

and

$$\varphi = \varphi_1 + \epsilon_2 \varphi_2(\xi,\eta,t), \tag{2.5}$$

and the perturbation equations linear in ϵ_2 are obtained as

$$w_{2,t} + J(\beta w_{2,\xi} - \varphi_{2,\eta}) = 0 \quad (\eta = 0); \tag{2.6}$$

$$\varphi_{2,t} - J\beta\varphi_{2,\xi} + \tfrac{1}{2}\beta^2 J_\eta w_2 + y_\eta w_2 = 0 \quad (\eta = 0); \tag{2.7}$$

$$\varphi_{2,\eta} \to 0 \quad \text{for} \quad \eta \to -\infty; \tag{2.8}$$

and

$$\varphi_{2,\xi\xi} + \varphi_{2,\eta\eta} = 0; \tag{2.9}$$

where

$$J = \left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right| \tag{2.10}$$

(see also Longuet-Higgins 1978*a*, *b*).

The coupling of the perturbation to the Stokes wave is thus completely described by the three 2π-periodic functions

$$J(\xi,\eta=0), \quad J_\eta(\xi,\eta=0), \quad y_\eta(\xi,\eta=0). \tag{2.11}$$

This is the principal advantage of our transformation.

We shall seek eigensolutions of the form ($\sigma_2 = k_2^{\frac{1}{2}}$),

$$w_2(\xi,t) = \exp[i(k_2\xi - \omega_2 t)]\,F(\xi) \tag{2.12}$$

and

$$\varphi_2(\xi,\eta,t) = \exp[i(k_2\xi - \omega_2 t)]\exp(|k_2|\eta)\,\sigma_2^{-1}\,G(\xi,\eta) \tag{2.13}$$

with

$$F(\xi) = F(\xi + 2\pi) \tag{2.14}$$

and

$$G(\xi,\eta) = G(\xi + 2\pi, \eta). \tag{2.15}$$

We consider only $k_2 > 0$, since the case $k_2 < 0$ can be obtained by conjugation.

From (2.8) and (2.9) we have

$$\exp(|k_2|\eta)\,G = \sum_{n=-\infty}^{+\infty} B_n \exp(in\xi)\exp(|n+k_2|\eta). \tag{2.16}$$

We introduce EG by
$$G_\eta = -iG_\xi + EG \quad (\eta = 0). \tag{2.17}$$
Inserting (2.12) and (2.13) into (2.6) and (2.7) we obtain, except for the term EG, a second-order ordinary differential equation with periodic coefficients for
$$G(\xi) = G(\xi, \eta = 0).$$
By standard procedures (see the appendix) this is reduced to a variant of Hill's equation
$$G_2'' + a_2(u)\, G_2(u) = E_2 G_2, \tag{2.18}$$
where $u = 2\xi$ and $a_2(u) = a_2(u; \alpha, \epsilon, \Lambda)$ with
$$\Lambda = (\omega_2 + k_2\beta)/\sigma_2 = \Omega_2/\sigma_2 \tag{2.19}$$
and
$$a_2(u) = a_2(-u) = a_2(u+\pi). \tag{2.20}$$
Further with $G_1(u) = G(\xi)$
$$G_2(u) = \exp(-i\mu u)\, G_1(u)\, P(u), \tag{2.21}$$
where
$$\mu = \mu(\alpha, \epsilon, \Lambda) \tag{2.22}$$
and $P(u) = P(u+\pi)$.
Finally $E_2 G_2$ is a transformed form of EG and will be neglected (see § 4).

The condition $G_1(u) = G_1(u+\pi)$ determines Λ since the characteristic exponent ν can be determined from (2.18), thus with (2.22)
$$\nu(\alpha, \epsilon, \Lambda) + \mu(\alpha, \epsilon, \Lambda) = 2n. \tag{2.23}$$
The theory is easily evaluated $O(\epsilon^2)$ and results will only be presented near the exceptional points, since only here do we find deviations from WKBJ results. These asymptotics $\epsilon \to 0$ may be quite misleading at values of, say, $\epsilon \approx 0{\cdot}4$, where $J(\xi, \eta = 0)$ nearly vanishes at $\xi = 0$.

3. Eigenvalues and eigensolutions

To second order
$$\mu = (1 - 2\Lambda/\alpha) + \epsilon^2(2/\alpha^2 - \Lambda/\alpha) + O(\epsilon^3) \tag{3.1}$$
and
$$a_2(u) = a - 2q_1 \cos 2u - 2q_2 \cos 4u + O(\epsilon^3), \tag{3.2}$$
where
$$a = (2/\alpha)^2\{(1 - \alpha\Lambda + \tfrac{1}{4}\alpha^2)(1 + 4\epsilon^2) - \tfrac{1}{2}\alpha\epsilon^2(\Lambda - \alpha)\} + O(\epsilon^3), \tag{3.3}$$
$$q_1 = -2\epsilon(a-1) + O(\epsilon^3) \tag{3.4}$$
and $q_2 = O(\epsilon^2)$ is not needed, unless $\sqrt{a} \approx 2$ or $m = 2$. We shall here consider $m \geqslant 3$ only. At $\epsilon = 0$ we have $\nu = \pm\sqrt{a}$ which yields eigenvalues
$$\Lambda_n^{(\pm)} = \{\pm(k_2+n)^{\frac{1}{2}} - (k_2+n) + k_2\}/k_2^{\frac{1}{2}} \tag{3.5}$$
corresponding to wavenumbers $\tilde{k}_2 = k_2 + n$, see (2.19). (Due to E_2G_2 (3.5) cannot be extended to $k_2 + n < 0$. In this case consider $\hat{k}_2 = N - \Delta k$, where $k_2 = N + \Delta k$ and $0 \leqslant \Delta k < 1$.) The exceptional points lie at $\nu = m$, $\sqrt{a} = m$ and for $\epsilon \neq 0$ we expand around these, $\nu = m + \Delta\nu$, $\mu = m + \Delta\mu$, $a = m^2 + \Delta a$. With a standard approximation for $\cos \pi\nu$ (Meixner & Schäfke 1954, p. 124) we obtain
$$\Delta\nu = \pm\frac{1}{2m}\{(a - A)^2 - \delta^2\}^{\frac{1}{2}} \tag{3.6}$$

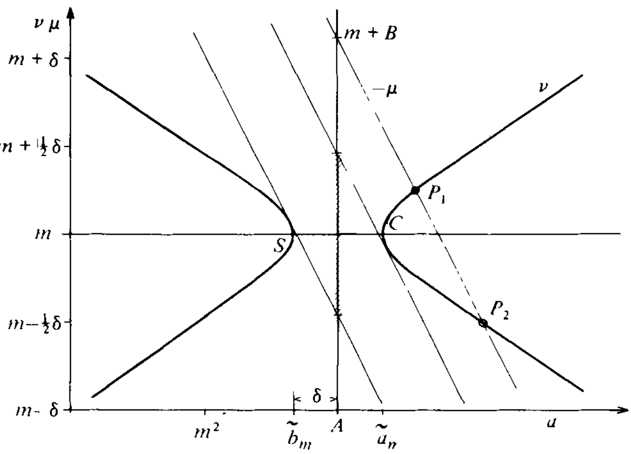

FIGURE 1. The μ, ν dispersion relation near resonance ($m = 3$). With ν and μ given by (3.6) and (3.7), B by (3.10) and $a-A = -4m\Delta\sigma_2 - 4(\Omega_2-\sigma_2) + 2\epsilon^2(m^2-\frac{1}{2}m+\frac{3}{2})$ the intersections P_1 and P_2 determine $\Omega_2-\sigma_2$ for given σ_2 and ϵ. For $\epsilon \to 0$, P_1 is associated with the forward propagating wave at k_2 while P_2 corresponds to the backward travelling wave at k_2-m. With increasing ϵ coupling yields instability $O(\epsilon^m)$ if B lies in the shaded interval of length $\delta = O(\epsilon^m)$. At the tangent points near C and S the unstable solutions show cos $\frac{1}{2}mx$ and sin $\frac{1}{2}mx$ amplitude modulation.

and
$$-\Delta\mu = -\tfrac{1}{2}(a-A)+B, \tag{3.7}$$
where
$$A = \tfrac{1}{2}(\tilde{a}_m+\tilde{b}_m) = 2(m^2-1)\epsilon^2+m^2+O(\epsilon^4), \tag{3.8}$$
$$\delta = \tfrac{1}{2}(\tilde{a}_m-\tilde{b}_m) = O(\epsilon^m), \tag{3.9}$$
$$B = \tfrac{1}{2}\epsilon^2(m-1)^2-2(m+1)\Delta\sigma_2, \tag{3.10}$$
$$\Delta\sigma_2 = \tfrac{1}{2}(m+1)-\sigma_2 \quad \text{and} \quad \sigma_2 = k^{\frac{1}{2}}_{\frac{1}{2}}. \tag{3.11}$$

While (3.6) with (3.9) have been strictly derived only for the Mathieu equation, with $\tilde{a}_m = \tilde{a}_m(\epsilon)$ given by the intersection of $a = a_m(q_1)$ with $q_1 = \epsilon(a-1)$ – in standard notation for Mathieu's equation – to the order given the same solutions (for $m \geqslant 3$) will hold for Hill's equation, except that δ in (3.9) could possibly be smaller, $\delta = O(\epsilon^N)$, $N > m$. While possible, this would require a delicate balancing of higher-order terms and would in view of what is said in the introduction appear as an unlikely coincidence.

The equation
$$\Delta\nu+\Delta\mu = 0 \tag{3.12}$$
is sketched in figure 1 for $m = 3$.

At P_1 and $\epsilon = 0$ we have $\Lambda_1 = 1$, $k_2 = 1/\alpha^2$ and at P_2, $\tilde{\Lambda}_2 = -1$, $\tilde{k}_2 = k_2-m$, where $\tilde{\Lambda}_2$ indicates that $\tilde{k}_2$ instead of k_2 has been used in (2.19). The eigensolutions for $\epsilon \neq 0$ at P_1 and P_2 are obtained as
$$G_2(u) = me_{\nu_1}(u;\, q_1(P_1)),$$
$$\nu_1 = m+\Delta\nu_1, \quad \nu_1+\mu = 0 \quad \text{at} \quad P_1, \tag{3.13}$$
and as
$$G_2(u) = me_{-(m-\Delta\nu_2)}(u) = me_{m-\Delta\nu_2}(-u),$$
$$\nu_2 = m+\Delta\nu_2-2m, \quad \nu_2+\mu = -2m, \quad \Delta\nu_2 < 0 \quad \text{at} \quad P_2. \tag{3.14}$$

For Hill's equation we should replace the me_ν by $he_\nu^s(u, \epsilon)$ the solutions of our particular Hill (Stokes) equation. For the Λ values we find (valid also away from the exceptional points)

$$\Lambda_1 = 1 + \frac{\epsilon^2}{\alpha}\{1 + O(1/\alpha)\} \tag{3.15}$$

and at P_2

$$\tilde{\Lambda}_2 = -1 + \frac{\epsilon^2}{\alpha}\{1 + O(1/\alpha)\} \tag{3.16}$$

in agreement with results obtained by Longuet-Higgins & Phillips (1962) for $\epsilon \ll \alpha^2$, and recently numerically confirmed [Longuet Higgins 1978*a*, see his equation (7.4)]. For $|B| < B_i = \frac{1}{2}\delta\,(1 - 1/m^2)^{\frac{1}{2}}$ or

$$|\Delta\sigma_2 - d(\epsilon)| = \left|\Delta\sigma_2 - \frac{\epsilon^2}{4}\frac{(m-1)^2}{m+1}\right| < B_i = O(\epsilon^m), \tag{3.17}$$

Λ becomes complex and we have instability. At the onset of instability $|B| = B_i$, the eigensolutions and eigenvalues at P_1 and P_2 are identical in the limit $P_1 \to P_2$ and a second linearly growing solution appears. Associated with the instability is a very rapid change in the eigensolutions: The $me_\nu u$ pass from $me_\nu u \approx e^{i\nu u}$ for $|B| - B_i \gg \delta$, over $2^{\frac{1}{2}} ce_m u$ for $P_1 = C$ to $-i2^{\frac{1}{2}} se_m u$ for $P_2 = S$, back to $e^{i\nu u}$ for $B < 0\,(|B| - B_i) \gg \delta$.

A uniformly valid approximation describing quantitatively how the me_ν pick up the Bragg-scattered component as $\Delta\sigma_2 - d(\epsilon)$ changes sign has not been obtained, but the considerations above suggest that the Bragg-scattered component is large only in a range

$$\Delta\sigma_2 = d(\epsilon) \pm 2\delta, \tag{3.18}$$

where

$$\delta = O(\epsilon^m). \tag{3.19}$$

4. The influence of the term E

We cannot say much about the effect of E, except that its effect should be negligible if our solutions have negligible contributions at negative wavenumbers.

Clearly E is an important term in the Benjamin–Feir instability at $m = 1$. For $m > 1$ we assume that our eigensolutions have Fourier amplitudes of unity at

$$k = \tfrac{1}{4}(m+1)^2 \quad \text{and} \quad k = \tfrac{1}{4}(m-1)^2$$

tapering off by one power of ϵ for each coupling step to either side. (This is a crude description for $m^2\epsilon \gg 1$, but in this range the term E is certainly negligible.) If then the first negative wavenumber reached by coupling has a Fourier component $O(\epsilon^n)$ the feedback into $k = \frac{1}{4}(m-1)^2$ will be $O(\epsilon^{2n})$ and our stability analysis not obviously inconsistent with our approximation $E = 0$ if $2n > m$. At $m = 2$ we find $n = 1$ and thus we expect our analysis to be qualitatively but not quantitatively correct, since not all terms $O(\epsilon^2)$ are included. At $m = 3$ we find $n = 3$ and thus we expect our treatment to be asymptotically correct for $\epsilon \to 0$ and $m \geq 3$, or $\alpha < 2/(m+1) = \frac{1}{2}$.

5. Comparison with the numerical results of Longuet-Higgins (1978*a*)

The agreement between the analytical and numerical results for Λ has already been discussed by Longuet-Higgins. Except for one pair all of his calculations lie well outside the regions (3.18), and his eigenfunctions behave according to WKBJ predictions. Of particular interest is the pair at ($k_2 = 4$, $\Omega > 0$) and ($k_2 = 1$, $\Omega < 0$) ($n = 4$ and $n = -1$ in Longuet-Higgins' notation), associated with $m = 3$. Neither at $n = -1$ nor at $n = 4$ is there any indication of instability in the computed eigenfrequencies or any noticeable admixture of the Bragg scattered component in the eigensolutions at $\epsilon = 0{\cdot}2$. While a strong contribution from $n = -1$ might be difficult to see at $n = 4$, a strong component of $n = 4$ to the solution at $n = -1$ should be easily detectable. From (3.18) and (3.19) this result is not in disagreement with our admittedly vague estimates. With $\Delta\sigma_2 = 0$, the relevant difference is

$$|\Delta\sigma_2 - d(\epsilon)| = \tfrac{1}{4}\epsilon^2 \tag{5.1}$$

and is presumably large compared to

$$\delta = O(\epsilon^3). \tag{5.2}$$

These vague comparisons clearly show the need to obtain a quantitative grasp on δ. Until this has been obtained a conceivable way of testing the theory would be to run several computations at a fixed ϵ, in fact a smaller value, say $\epsilon = 0{\cdot}1$, would be more convenient, and to cover the range, say near $m = 3$,

$$\sigma_2 = 2 - \tfrac{1}{4}\epsilon^2 \pm O(\epsilon^3). \tag{5.3}$$

The detection of the instability should be possible in a numerical calculation, but even more striking should be the rapid change in the eigenfunctions, which could best be demonstrated by plotting amplitude and phase of the Fourier components at $k_n = k_2 + n$ against n.

Alternatively one could hold σ_2 fixed at $2 - \frac{1}{4}\epsilon_0^2$ for $\epsilon_0 = 0{\cdot}1$ and then vary ϵ.

If any of these calculations fail to produce the predicted features our theory has clearly failed the test.

6. Summary and discussion

It has been shown that the stability analysis of perturbations on a Stokes wave can be reduced to an equation

$$G_2'' + a_2(u, \Lambda, \epsilon, \alpha)\, G_2 = E_2\, G_2, \tag{6.1}$$

where $a_2(u, \Lambda)$ is an even π-periodic function of u and a linear function of the eigenvalue Λ. E_2 is a complicated operator, but its influence decreases rapidly with increasing wavenumber of the perturbation.

For small values of ϵ (6.1) may be reduced to the Mathieu equation and regions of instability for wavenumbers close to $k_2 = \frac{1}{4}(m+1)^2$ or $a_2 \approx m^2$ are predicted. However, due to the influence of the term E_2 the two strongest instabilities $O(\epsilon^2)$ at $m = 1$ and $m = 2$ are not correctly treated, with qualitatively incorrect results at $m = 1$ and quantitatively incorrect results at $m = 2$. For $m \geqslant 3$ we have provided only order of

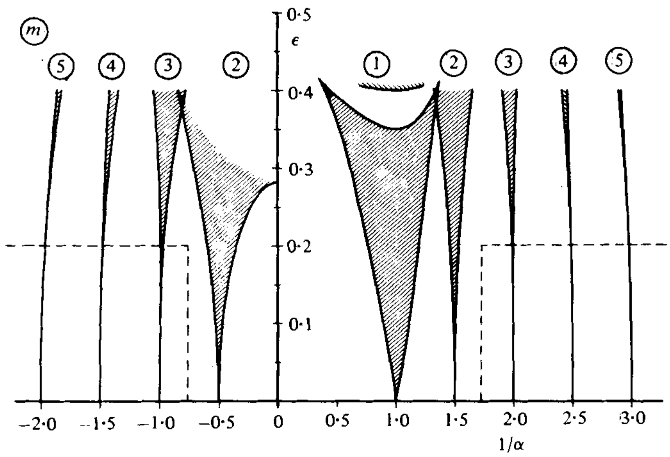

FIGURE 2. Stability diagram for perturbations of a Stokes wave (at $\alpha = 1$), negative values of $\alpha^{-1} = \pm(|k_2|/k_1)^{\frac{1}{2}}$ corresponding to backward running waves. In the shaded area we have instability. The areas at $m = 1$ are taken from Longuet-Higgins (1978*b*, figure 6). For $m > 1$ the stability diagram is qualitative at best outside of the area enclosed by the broken line, which marks the approximate limits of validity of our treatment. Further details are explained in the summary.

magnitude results for the strength and location of the instability, predicting instability for

$$|k_2^{\frac{1}{2}} - [\tfrac{1}{2}(m+1) - \tfrac{1}{4}\epsilon^2(m-1)^2/(m+1)] + O(\epsilon^3)| = O(\epsilon^m); \tag{6.2}$$

thus for $m > 3$ the width is better known than the centre of the unstable region. The frequency of the component k_2 in the system R_D (in which the water at depth is at rest) is

$$\Omega_2 = k_2^{\frac{1}{2}}(1 + \epsilon^2(k_2^{\frac{1}{2}} + O(k_2^{-\frac{1}{2}})) + ik_2^{\frac{1}{2}}\,O(\epsilon^m). \tag{6.3}$$

Due to Bragg scattering the unstable waves consist of one forward travelling component at k_2 and a backward travelling component at $k_2 - m$, so that both components have the same frequency in the system R_s in which the Stokes wave is at rest. The two components are in phase at the lower edge of the unstable region (6.2) and in antiphase at the upper edge. Thus in R_s the waves on the boundaries of the stable regions have forms close to

$$w_2(x,t) = t.\exp[i\tfrac{1}{4}(m^2+1)\,x - i\omega_2 t].\begin{Bmatrix}\cos\frac{1}{2}mx\\ \sin\frac{1}{2}mx\end{Bmatrix}\exp[i\epsilon k_2 \sin x]. \tag{6.4}$$

The last factor in (6.4) describes wavenumber modulation. Inside the unstable regions, where we have exponential growth, the waveform is not well understood. If we are well away from the unstable regions (6.2) we recover the WKBJ results.

The areas of instability shown in figure 2 have been constructed from (6.2) with the term $O(\epsilon^3) = 0$ and the term $O(\epsilon^m) = \epsilon^m$. At $m = 2$ this is incorrect $O(\epsilon^2)$ (see the remarks after (3.4) and in § 4), and, while the qualitative nature of a width $O(\epsilon^2)$ is preserved, the extension of the instability zone to $k = 0$ for $m = 2$, is probably an incorrect feature. The instability area for each m would be symmetric in a k_2 instead of the $\pm|k_2|^{\frac{1}{2}}$ presentation, the area at $k_2 \approx \frac{1}{4}(m+1)^2$ corresponding to an area at

$\bar{k}_2 = -(k_2 - m)$. The instability areas have been extended to larger values of ϵ than covered by the theory in order to show some width at larger m. Obviously interesting questions concerning the dependence of the instability areas on ϵ for larger values of ϵ and on the angle of incidence arise.

We suggest that numerical computations be concentrated near (6.2), both to obtain a test of the theory and also to stay in the areas of most interest. It is possible that the higher-order instabilities become more important with increasing ϵ. Only the term E_2 has prevented a unified analytical treatment of all resonances at $m = 1, 2 \ldots$, and has thus not allowed us to see whether these resonances share common features.

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Appendix

From (2.6) and (2.7)

$$F = iR\{N + \gamma Q\}\, G, \tag{A 1}$$

$$MG = -i\{N + \gamma Q\}\, F = \{N + \gamma Q\}\, R\{N + \gamma Q\}\, G, \tag{A 2}$$

where

$$M = 1 + \alpha^2 \partial_\eta, \quad N = \Lambda - i(\alpha\beta)\,\partial_\xi, \tag{A 3a, b}$$

$$\gamma = \omega_2 \alpha = \Lambda - \beta/\alpha, \tag{A 4}$$

$$Q = (1 - J)/J \tag{A 5}$$

and

$$R = J \bigg/ \left(y_\eta + \frac{\beta^2}{2} J_\eta\right). \tag{A 6}$$

With $R_1(u) = R(\xi)$, $Q_1(u) = Q(\xi)$,

$$\theta(u) = \frac{2}{\alpha\beta} \int^u \left\{\frac{\alpha}{2\beta} \frac{1}{R_1(u')} - (\Lambda + \gamma Q_1)\right\} du' \tag{A 7}$$

and

$$G_2(u) = G_1(u) \exp\left[-i\theta(u)\right] R_1^{-\frac{1}{2}}, \tag{A 8}$$

we transform (A 2) to (2.18) (Bellman 1964), where

$$a_2(u) = \frac{2}{(\alpha\beta)^2} \frac{1}{R_1(u)} \left\{1 + \frac{(\alpha\beta)^2}{16} \left(\frac{R_1'}{R_1}\right)^2 - \tfrac{1}{8}(\alpha\beta)^2 R_1'' + \left(\frac{\alpha}{2\beta}\right)^2 \frac{1}{R_1} - \frac{\alpha}{\beta}(\Lambda + \gamma Q_1)\right\} \tag{A 9}$$

and

$$E_2 G_2 = \frac{8}{\beta^2} \exp\left[-i\theta(u)\right] \frac{1}{R_1^{\frac{3}{2}}} E_1(\alpha)\, G_1, \tag{A 10}$$

where $E_1(\alpha)$ is the operator which maps

$$f(u) = \sum_n f_n\, e^{i\lambda u}\, e^{2inu} \tag{A 11}$$

onto

$$E_1 f = \sum \hat{f}_n\, e^{i\lambda u}\, e^{2inu}, \tag{A 12}$$

$$\hat{f}_n = \begin{cases} (\frac{1}{2}\lambda + n) f_n & \text{for} \quad \frac{1}{2}\lambda + n < -1/\alpha^2, \\ 0 & \text{otherwise.} \end{cases} \tag{A 13}$$

To second order
$$R_1(u) = 1-2\epsilon\cos 2u - 2\epsilon^2\cos 4u + O(\epsilon^3), \tag{A 14}$$
$$Q_1(u) = 2\epsilon\cos 2u + 4\epsilon^2\cos 4u + \epsilon^2 + O(\epsilon^3), \tag{A 15}$$
$$\theta(u) = \mu u + \theta_1\epsilon\sin 2u + \theta_2\epsilon^2\sin 4u + O(\epsilon^3), \tag{A 16}$$
$$\theta_1 = \frac{2}{\alpha}\left(\gamma - \frac{\alpha}{2}\right) + O(\epsilon^2) \tag{A 17}$$
and
$$\theta_2 = -\frac{2}{\alpha^2}\left(1-\Lambda\alpha+\frac{\alpha^2}{2}\right) + O(\epsilon), \tag{A 18}$$
and in (3.2)
$$q_2 = -\epsilon^2(5a-7) + O(\epsilon^3). \tag{A 19}$$

For Hill's equation see books (Arscott 1964; Ince 1926; Whittaker & Watson 1927).

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